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A NOTE ON THE (ANTI-)BRST INVARIANT LAGRANGIAN DENSITIES FOR THE FREE ABELIAN 2-FORM GAUGE THEORY

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Abstract: We show that the previously known off-shell nilpotent ($s_{(a)b}^2 = 0$) and absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) Becchi-Rouet-Stora-Tyutin (BRST) transformations (s_b) and anti-BRST transformations (s_{ab}) are the symmetry transformations of the appropriate Lagrangian densities of a four (3 + 1)-dimensional (4D) free Abelian 2-form gauge theory which do *not* explicitly incorporate a very specific constrained field condition through a Lagrange multiplier 4D vector field. The above condition, which is the analogue of the Curci-Ferrari restriction of the non-Abelian 1-form gauge theory, emerges from the Euler-Lagrange equations of motion of our present theory and ensures the *absolute* anticommutativity of the transformations $s_{(a)b}$. Thus, the coupled Lagrangian densities, proposed in our present investigation, are aesthetically more appealing and more economical.

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1 Introduction

The principle of local gauge invariance provides a precise theoretical basis for the description of the three (out of four) fundamental interactions of nature. The theories with local gauge symmetries are always (i) described by the singular Lagrangian densities, and (ii) endowed with the first-class constraints in the language of Dirac's prescription for the classification scheme [1,2]. It has been well-established that the latter (i.e. the first-class constraints) generate the above local gauge symmetry transformations for the singular Lagrangian densities of the relevant gauge theories.

One of the most attractive approaches to covariantly quantize such kind of theories is the BRST formalism where (i) the unitarity and "quantum" gauge (i.e. BRST) invariance are respected together, (ii) the true physical states are defined in terms of the BRST charge which turn out to be consistent with the Dirac's prescription for the quantization of systems with constraints, and (iii) there exists a deep relationship between the physics of the gauge theories (in the framework of BRST formalism) and the mathematics of differential geometry (e.g. cohomology) and supersymmetry (e.g. superfield formalism).

Some of the key and cute mathematical properties, associated with the (anti-)BRST symmetry transformations, are as follows. First, there exist two symmetry transformations (christened as the (anti-)BRST[1] symmetry transformations $s_{(a)b}$) for a given local gauge symmetry transformation. Second, both the symmetries are nilpotent of order two (i.e. $s_{(a)b}^2 = 0$). Finally, they anticommute (i.e. $s_b s_{ab} + s_{ab} s_b = 0$) with each-other when they act *together* on any specific field of the theory. These properties are very sacrosanct for any arbitrary gauge (or reparametrization) invariant theory when the latter is described within the framework of the BRST formalism.

Recently, the 2-form ($B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu) B_{\mu\nu}$) Abelian gauge field $B_{\mu\nu}$ [6,7] and corresponding gauge theory have attracted a great deal of interest because of their relevance in the context of (super)string theories. This Abelian 2-form gauge theory has also been shown to provide (i) an explicit field theoretical example of the Hodge theory [8], and (ii) a model for the quasi-topological field theory [9]. The (anti-)BRST invariant Lagrangian densities of the 2-form theory have been written out and the BRST quantization has been performed [8-12]. One of the key observations in (see, e.g. [8,9]) is that the above (anti-)BRST transformations, even though precisely off-shell nilpotent, are found to be anticommuting only up to a vector gauge transformation. Thus, the absolute anticommutativity property is lost.

[1] We follow here the standard notations and conventions adopted in our recent works on 4D free Abelian 2-form gauge theory within the framework of BRST formalism [3-5].

As pointed out earlier, the anticommutativity property of the (anti-)BRST symmetry transformations is a cardinal requirement in the domain of application of the BRST formalism to gauge theories. This key property actually encodes the linear independence of the above two transformations corresponding to a given local gauge symmetry transformation (of a specific gauge theory). In the realm of superfield approach to BRST formalism (see, e.g., [13,5]), the absolute anticommutativity of these transformations becomes crystal clear because these are identified with the translational generators along the Grassmannian directions of a (D, 2)-dimensional supermanifold on which any arbitrary D-dimensional gauge theory is considered [13,5].

It is worthwhile to mention that, the superfield approach, proposed in [13] for the 4D non-Abelian 1-form gauge theory, has been applied, for the first time, to the description of the free 4D Abelian 2-form gauge theory in [5]. One of the upshots (of the discussions in [5]) is that an analogue of the Curci-Ferrari (CF) type of restriction [14] emerges in the context of 4D *Abelian* 2-form gauge theory. The former happens to be the hallmark of a 4D *non-Abelian* 1-form gauge theory [13,14]. This CF type condition ensures (i) the absolute anticommutativity of the (anti-)BRST symmetry transformations of the Abelian 2-form gauge theory, and (ii) the identification of the (anti-) BRST symmetry transformations with the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold [5].

Keeping the above properties in mind, the (anti-)BRST symmetry transformations have been obtained in our earlier works [3,4] where the above CF type field condition is invoked for the proof of the absolute anticommutativity of the off-shell nilpotent (anti-)BRST symmetry transformations [3,4]. In fact, the above field condition is explicitly incorporated in the Lagrangian densities through a Lagrange multiplier vector field (which is not a basic dynamical field of the theory). Furthermore, due to the above restriction, the kinetic term for the massless scalar field of the theory turns out to possess a negative sign. These are the prices one pays to obtain the absolute anticommutativity of the nilpotent (anti-)BRST symmetry transformations.

The purpose of our present investigation is to show that the (anti-)BRST tansformations of our earlier works [3,4] are the *symmetry* transformations of a pair of coupled Lagrangian densities which do not incorporate the analogue of the CF type restriction explicitly through the Lagrange multiplier 4D vector field[2]. This condition, however, appears in the theory as a consequence of the Euler-Lagrange equations of motion that are derived from the coupled Lagrangian densities. Furthermore, all the terms of these Lagrangian

[2]This feature is exactly like the discussion of the absolutely anticommuting (anti-)BRST symmetry transformations in the context of the 4D non-Abelian 1-form gauge theory.

densities carry standard meaning and there are no peculiar signs associated with any of them. One of the key features of the CF type restriction, for our present Abelian theory, is that it does not involve any kind of (anti-)ghost fields. On the contrary, one knows that the original CF restriction of the non-Abelian 1-form gauge theory [14] does involve the (anti-)ghost fields.

The key factors that have propelled us to pursue our present investigation are as follows. First and foremost, it is very important to obtain the correct nilpotent and anticommuting (anti-)BRST symmetry transformations which are respected by the appropriate Lagrangian densities. The latter should, for aesthetic reasons, be economical and beautiful (i.e. possessing no peculiar looking terms). Second, the theory itself should produce all the cardinal requirements and nothing should be imposed from outside through a Lagrange multiplier field. Third, the (anti-)BRST symmetry transformations in the Lagrangian formulation [3,4] must be consistent with the derivation of the same from the superfield approach [5]. Finally, our present study is the first modest step towards our main goal of applying the BRST formalism to higher p-form ($p > 2$) gauge theories that are relevant in (super)string theories.

The contents of our present investigation are organized as follows. In Sec. 2, we briefly recapitulate the bare essentials of the off-shell nilpotent and anticommuting (anti-)BRST symmetry transformations for a couple of Lagrangian densities of the Abelian 2-form gauge theory. The above Lagrangian densities incorporate a constrained field relationship through a Lagrange multiplier 4D vector field. Our Sec. 3 deals with a pair of coupled and equivalent Lagrangian densities that (i) respect the BRST and anti-BRST symmetry transformations, and (ii) do not incorporate any constrained field relationship explicitly. In Sec. 4, we derive an explicit BRST algebra by exploiting the infinitesimal continuous symmetry transformations. We make some concluding remarks in our Sec. 5.

2 Preliminaries: Lagrangian Densities Incorporating the Constrained Field Condition

We begin with the following nilpotent (anti-)BRST symmetry invariant Lagrangian density for the 4D free Abelian 2-form gauge theory [3] [3-5]

$$\mathcal{L}^{(1)} \;=\; \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + B^{\mu}(\partial^{\nu} B_{\nu\mu}) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi$$

[3]We choose the 4D spacetime metric $\eta_{\mu\nu}$ with the signatures $(+1, -1, -1, -1)$ so that $P \cdot Q = \eta_{\mu\nu} P^{\mu} Q^{\nu} = P_0 Q_0 - P_i Q_i$ is the dot product between non-null four vectors P_{μ} and Q_{μ}. Here $\mu, \nu, \kappa, \sigma ... = 0, 1, 2, 3$ and $i, j, k .... = 1, 2, 3$. We also adopt, in the whole body of our text, the field differentiation convention: $(\delta B_{\mu\nu} / \delta B_{\kappa\sigma}) = \frac{1}{2!}(\delta_{\mu\kappa}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\kappa})$, etc.

$$
\begin{aligned}
&+ \quad \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho \\
&+ \quad (\partial \cdot \bar{C} + \rho)\lambda + L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi),
\end{aligned}
\tag{1}
$$

where the kinetic term is constructed with the totally antisymmetric curvature tensor $H_{\mu\nu\kappa}$ which is derived from the 3-form $H^{(3)} = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\kappa) H_{\mu\nu\kappa}$. The exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the 2-form $B^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) B_{\mu\nu}$ generate the above 3-form (i.e. $H^{(3)} = dB^{(2)}$).

We have the Lorentz vector fermionic (anti-)ghost fields $(\bar{C}_\mu) C_\mu$ and the bosonic (anti-)ghost fields $(\bar{\beta})\beta$ in the theory. The above Lagrangian density also requires fermionic auxiliary ghost fields $\rho = -\frac{1}{2}(\partial \cdot \bar{C})$ and $\lambda = \frac{1}{2}(\partial \cdot C)$. The auxiliary vector fields B_μ and $\bar{B}_\mu$ are constrained to satisfy the field equation $B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0$ where the massless (i.e. $\Box \phi = 0$) field ϕ is required for the stage-one reducibility in the theory. The above constrained field equation emerges due to presence of the Lagrange multiplier field L^μ.

The following off-shell nilpotent (i.e. $s_b^2 = 0$) BRST symmetry transformations s_b for the 4D local fields of the theory, namely;

$$
\begin{aligned}
&s_b B_{\mu\nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \qquad s_b C_\mu = -\partial_\mu \beta, \qquad s_b \bar{C}_\mu = -B_\mu, \\
&s_b L_\mu = -\partial_\mu \lambda, \qquad s_b \phi = \lambda, \qquad s_b \bar{\beta} = -\rho, \\
&s_b \bar{B}_\mu = -\partial_\mu \lambda, \qquad s_b [\rho, \lambda, \beta, B_\mu, H_{\mu\nu\kappa}] = 0,
\end{aligned}
\tag{2}
$$

leave the above Lagrangian density quasi-invariant because it transforms to a total spacetime derivative: $s_b \mathcal{L}^{(1)} = -\partial_\mu [(\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu + \lambda B^\mu + \rho \partial^\mu \beta]$.

In exactly similar fashion, the following off-shell nilpotent ($s_{ab}^2 = 0$) anti-BRST symmetry transformations s_{ab}

$$
\begin{aligned}
&s_{ab} B_{\mu\nu} = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = +\bar{B}_\mu, \\
&s_{ab} L_\mu = -\partial_\mu \rho, \qquad s_{ab} \phi = \rho, \qquad s_{ab} \beta = -\lambda, \\
&s_{ab} B_\mu = +\partial_\mu \rho, \qquad s_{ab} [\rho, \lambda, \bar{\beta}, \bar{B}_\mu, H_{\mu\nu\kappa}] = 0,
\end{aligned}
\tag{3}
$$

leave the following Lagrangian density

$$
\begin{aligned}
\mathcal{L}^{(2)} \;=\;& \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \bar{B}^\mu (\partial^\nu B_{\nu\mu}) + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
+& \quad \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho \\
+& \quad (\partial \cdot \bar{C} + \rho)\lambda + L^\mu (B_\mu - \bar{B}_\mu - \partial_\mu \phi),
\end{aligned}
\tag{4}
$$

quasi-invariant because it transforms to a total spacetime derivative as is evident from $s_{ab} \mathcal{L}^{(2)} = -\partial_\mu [(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - \rho \bar{B}^\mu + \lambda \partial^\mu \bar{\beta}]$. It is interesting to point out that both the Lagrangian densities (1) and (4) respect the off-shell nilpotent (anti-)BRST symmetry transformations (cf. (2) and (3)) on a constrained surface defined by a field equation (see, e.g. equation (5) below).

Both the above nilpotent transformations $s_{(a)b}$ (cf. (2) and (3)) are absolutely *anticommuting* (i.e. $s_b s_{ab} + s_{ab} s_b \equiv \{s_b, s_{ab}\} = 0$) in nature if the whole 4D free Abelian 2-form gauge theory is defined on a constrained surface parametrized by the following field equation [4]

$$B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0. \tag{5}$$

This is due to the fact that $\{s_b, s_{ab}\} B_{\mu\nu} = 0$ is true only if the above equation is satisfied. This condition has been incorporated in the above Lagrangian densities through the Lagrange multiplier Lorentz 4D vector field L^μ.

The Lagrangian densities (1) and (4) are coupled Lagrangian densities on the constrained field surface defined by (5). It would be very nice if one could obtain Lagrangian densities that respect the nilpotent and anticommuting (anti-)BRST symmetry transformations (2) and (3) and are free of any Lagrange multiplier field. The latter fields are required when we wish to put some restriction, from outside, on the theory. A beautiful theory should produce this restriction on its own strength. Thus, it is desired that the Lagrangian density of a theory should be devoid of Lagrange multipliers. Furthermore, it would be better if we could avoid the negative kinetic term for the massless scalar field ϕ that is present in the Lagrangian densities (1) and (4) of our present theory. We address these issues in our next section.

3 Lagrangian Densities Without Any Constrained Field Condition: Symmetries

It is interesting to note that the following coupled and equivalent (cf. (5)) Lagrangian densities for the 4D free Abelian 2-form gauge theory, namely;

$$\begin{aligned}
\mathcal{L}_B &= \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + B^\mu (\partial^\nu B_{\nu\mu} - \partial_\mu \phi) + B \cdot B + \partial_\mu \bar{\beta} \partial^\mu \beta \\
&+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\, \lambda,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathcal{L}_{\bar{B}} &= \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \bar{B}^\mu (\partial^\nu B_{\nu\mu} + \partial_\mu \phi) + \bar{B} \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^\mu \beta \\
&+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\, \lambda,
\end{aligned} \tag{7}$$

remain quasi invariant under the nilpotent and anticommuting (anti-)BRST symmetry transformations (2) and (3), respectively. However, these La-

[4]This restriction comes out from our previous work [5] that is devoted to the discussion of the free 4D Abelian 2-form gauge theory within the framework of superfield formalism.

grangian densities do not incorporate explicitly the constrained field condition (5). Neither do they possess negative kinetic term for the massless scalar field ϕ. Thus, above Lagrangian densities are the appropriate ones.

The above Lagrangian densities (6) and (7) are equivalent on the constrained surface (defined by the field equation (5)) because they respect both the BRST and anti-BRST symmetry transformations separately and independently. To clarify this statement explicitly, it can be checked that the Lagrangian density (6) transforms under the off-shell nilpotent (anti-)BRST symmetry transformations as given below

$$\begin{aligned} s_b \mathcal{L}_B &= s_b \mathcal{L}^{(1)}, \\ s_{ab} \mathcal{L}_B &= -\partial_\mu [(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu + \lambda \partial^\mu \bar{\beta} \\ &- \rho(\partial_\nu B^{\nu\mu} + \bar{B}^\mu)] + (B^\mu - \bar{B}^\mu - \partial^\mu \phi) \partial_\mu \rho \\ &+ \partial^\mu (B^\nu - \bar{B}^\nu - \partial^\nu \phi)(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu). \end{aligned} \tag{8}$$

In an exactly similar fashion, the Lagrangian density (7) changes under the (anti-)BRST symmetry transformations as

$$\begin{aligned} s_{ab} \mathcal{L}_{\bar{B}} &= s_{ab} \mathcal{L}^{(2)}, \\ s_b \mathcal{L}_{\bar{B}} &= -\partial_\mu [(\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{B}_\nu + \rho \partial^\mu \beta \\ &+ \lambda(\partial_\nu B^{\nu\mu} + B^\mu)] + (B^\mu - \bar{B}^\mu - \partial^\mu \phi) \partial_\mu \lambda \\ &- \partial^\mu (B^\nu - \bar{B}^\nu - \partial^\nu \phi)(\partial_\mu C_\nu - \partial_\nu C_\mu). \end{aligned} \tag{9}$$

Thus, on the constrained surface (defined by (5)), the Lagrangian densities (6) and (7) are equivalent and both of them respect the (anti-)BRST symmetry invariances. The condition (5), however, has to be imposed from outside.

The following Euler-Lagrange equations of motion

$$B_\mu = -\frac{1}{2}(\partial^\nu B_{\nu\mu} - \partial_\mu \phi), \qquad \bar{B}_\mu = -\frac{1}{2}(\partial^\nu B_{\nu\mu} + \partial_\mu \phi), \tag{10}$$

from the above Lagrangian densities (6) and (7) imply that

$$\begin{aligned} &\partial \cdot B = 0, \qquad \partial \cdot \bar{B} = 0, \qquad \Box \phi = 0, \\ &B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0, \qquad B_\mu + \bar{B}_\mu + \partial^\nu B_{\nu\mu} = 0. \end{aligned} \tag{11}$$

Thus, the analogue of the Curci-Ferrari restriction [14] of the non-Abelian 1-form gauge theory, is hidden in the above coupled Lagrangian densities in the form of the Euler-Lagrange equation of motion (cf. (11) *vis-à-vis* (5)).

To capture the above (anti-)BRST invariance in a simpler setting, it can be seen that the Lagrangian densities (6) and (7) can be re-expressed as the sum of the kinetic term and the BRST and anti-BRST exact forms, namely;

$$\mathcal{L}_B = \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + s_b \Big[-\bar{C}^\mu \{ (\partial^\nu B_{\nu\mu} - \partial_\mu \phi) + B_\mu \} + \bar{\beta}(\partial \cdot C - 2\lambda) \Big], \tag{12}$$

$$\mathcal{L}_{\bar{B}} = \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + s_{ab} \Big[+ C^\mu \{ (\partial^\nu B_{\nu\mu} + \partial_\mu \phi) + \bar{B}_\mu \} + \beta (\partial \cdot \bar{C} + 2\rho) \Big]. \quad (13)$$

The above equations provide a simple and straightforward proof for the nilpotent symmetry invariance of the Lagrangian densities (6) and (7) because of (i) the nilpotency (i.e. $s_{(a)b}^2 = 0$) of the transformations $s_{(a)b}$, and (ii) the invariance of the curvature term (i.e. $s_{(a)b} H_{\mu\nu\kappa} = 0$) under $s_{(a)b}$.

It will be noted that the following interesting expressions[5]

$$s_b s_{ab} \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big] = B^\mu (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^\mu \beta$$
$$+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda, \quad (14)$$

$$-s_{ab} s_b \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big] = \bar{B}^\mu (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^\mu \beta$$
$$+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda, \quad (15)$$

allow us to express the Lagrangian densities (6) and (7) in yet another forms

$$\mathcal{L}_B = \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + s_b s_{ab} \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big], \quad (16)$$

$$\mathcal{L}_{\bar{B}} = \frac{1}{6} H^{\mu\nu\kappa} H_{\mu\nu\kappa} - s_{ab} s_b \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big], \quad (17)$$

where one has to make use of (5) (or (11)) to express $(B \cdot \bar{B})$ either equal to $(B \cdot B - B^\mu \partial_\mu \phi)$ or equal to $(\bar{B} \cdot \bar{B} + \bar{B}^\mu \partial_\mu \phi)$. Once again, one can note the (anti-)BRST invariance of the Lagrangian densities (17) and (16) due to the nilpotency ($s_{(a)b}^2 = 0$) and invariance of the curvature term ($s_{(a)b} H_{\mu\nu\kappa} = 0$). It is worthwhile to mention that the Lagrangian densities in (1) and (4) cannot be recast into the forms like equations (12), (13), (16) and (17). The central obstacle in this attempt is created by the Lagrange multiplier term and kinetic term for the massless scalar field ϕ (cf. (1) and (4)).

The following global transformations of the fields

$$\begin{array}{llll} B_{\mu\nu} \to B_{\mu\nu}, & B_\mu \to B_\mu, & \bar{B}_\mu \to \bar{B}_\mu, & \phi \to \phi, \\ \beta \to e^{+2\Omega} \beta, & \bar{\beta} \to e^{-2\Omega} \bar{\beta}, & C_\mu \to e^{+\Omega} C_\mu, & \\ \bar{C}_\mu \to e^{-\Omega} \bar{C}_\mu, & \lambda \to e^{+\Omega} \lambda, & \rho \to e^{-\Omega} \rho, & \end{array} \quad (18)$$

(where Ω is an infinitesimal global parameter) leave the Lagrangian densities (6) and (7) invariant. A close look at the above transformations shows that

[5]These relations are similar to the case of non-Abelian 1-form gauge theory where the CF restriction is *not* explicitly incorporated in the Lagrangian densities (see, e.g., [13]).

all the ghost terms of (6) and (7) remain invariant under the above transformations. The infinitesimal version of the above global ghost transformations s_g (modulo parameter Ω) is such that $s_g \beta = 2\beta, s_g \bar{\beta} = -2\bar{\beta}, s_g C_\mu = +C_\mu, s_g \bar{C}_\mu = -\bar{C}_\mu, s_g \lambda = +\lambda, s_g \rho = -\rho$. The factors of ± 2 and ± 1, present in the exponentials of equation (18), correspond to the ghost numbers of the corresponding ghost fields which would play very significant roles in the next section where we shall compute some commutators with the ghost charge.

4 Generators of the Continuous Symmetry Transformations: BRST Algebra

The nilpotent (anti-)BRST symmetry transformations (3) and (2) and the infinitesimal version of the global transformations in (18) lead to the derivation of the Noether conserved currents. These are as follows

$$\begin{aligned} J^\mu_{(ab)} &= \rho \bar{B}^\mu - (\partial^\mu C^\nu - \partial^\nu C^\mu)\partial_\nu \bar{\beta} - H^{\mu\nu\kappa}(\partial_\nu \bar{C}_\kappa - \partial_\kappa \bar{C}_\nu) \\ &- \lambda \partial^\mu \bar{\beta} - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)\bar{B}_\nu, \\ J^\mu_{(b)} &= (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)\partial_\nu \beta - H^{\mu\nu\kappa}(\partial_\nu C_\kappa - \partial_\kappa C_\nu) \\ &- \rho \partial^\mu \beta - \lambda B^\mu - (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu, \\ J^\mu_{(g)} &= 2\beta \partial^\mu \bar{\beta} - 2\bar{\beta}\partial^\mu \beta + \lambda \bar{C}^\mu - \rho C^\mu \\ &+ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) C_\nu + (\partial^\mu C^\nu - \partial^\nu C^\mu)\bar{C}_\nu. \end{aligned} \tag{19}$$

It is straightforward to check that the continuity equation $\partial_\mu J^\mu_{(i)} = 0$ (with $i = b, ab, g$) is satisfied if we exploit the Euler-Lagrange equations of motion derived from the Lagrangian densities (6) and (7).

The above Noether conserved currents lead to the definition of the conserved and nilpotent ($Q^2_{(a)b} = 0$) (anti-)BRST charges ($Q_{(a)b} = \int d^3x J^0_{(a)b}$) and the conserved ghost charge ($Q_g = \int d^3x J^0_{(g)}$) as given below

$$\begin{aligned} Q_{ab} &= \int d^3x \Big[\rho \bar{B}^0 - \lambda \partial^0 \bar{\beta} - H^{0ij}(\partial_i \bar{C}_j - \partial_j \bar{C}_i) \\ &- (\partial^0 \bar{C}^i - \partial^i \bar{C}^0)\bar{B}_i - (\partial^0 C^i - \partial^i C^0)\partial_i \bar{\beta}\Big], \\ Q_b &= \int d^3x \Big[(\partial^0 \bar{C}^i - \partial^i \bar{C}^0)\partial_i \beta - H^{0ij}(\partial_i C_j - \partial_j C_i) \\ &- (\partial^0 C^i - \partial^i C^0) B_i - \lambda B^0 - \rho \partial^0 \beta\Big], \\ Q_g &= \int d^3x \Big[2\beta \partial^0 \bar{\beta} - 2\bar{\beta}\partial^0 \beta + (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) C_i \end{aligned}$$

$$- \quad \rho C^0 + \lambda \bar{C}^0 + (\partial^0 C^i - \partial^i C^0)\bar{C}_i\Big]. \tag{20}$$

These conserved charges $Q_{(a)b}$ and Q_g obey the following BRST algebra

$$\begin{aligned} &Q_b^2 = \frac{1}{2}\{Q_b, Q_b\} = 0, \qquad Q_{ab}^2 = \frac{1}{2}\{Q_{ab}, Q_{ab}\} = 0, \\ &Q_b Q_{ab} + Q_{ab} Q_b \equiv \{Q_b, Q_{ab}\} = 0 \equiv \{Q_{ab}, Q_b\}, \\ &i[Q_g, Q_b] = +Q_b, \qquad i[Q_g, Q_{ab}] = -Q_{ab}. \end{aligned} \tag{21}$$

The above algebra plays a key role in the cohomological description of the states of the quantum gauge theory in the quantum Hilbert space (QHS).

The algebra in (21) can be derived by exploiting the infinitesimal transformations $s_{(a)b}$ and s_g and the expressions for $Q_{(a)b}$ and Q_g. These are

$$\begin{aligned} &s_b Q_b = -i\{Q_b, Q_b\} = 0, \qquad s_{ab} Q_{ab} = -i\{Q_{ab}, Q_{ab}\} = 0, \\ &s_b Q_{ab} = -i\{Q_{ab}, Q_b\} = 0, \qquad s_{ab} Q_b = -i\{Q_b, Q_{ab}\} = 0, \\ &s_g Q_{ab} = -i[Q_{ab}, Q_g] = -Q_{ab}, \qquad s_g Q_b = -i[Q_b, Q_g] = Q_b, \\ &s_b Q_g = -i[Q_g, Q_b] = -Q_b, \qquad s_{ab} Q_g = -i[Q_g, Q_{ab}] = Q_{ab}. \end{aligned} \tag{22}$$

In the above computations, the factors of ± 2 and ± 1 present in the ghost transformations (18), play a very crucial role. Furthermore, some of the computations in the above are really non-trivial and algebraically more involved. In particular, in the proof of $\{Q_b, Q_{ab}\} \equiv \{Q_{ab}, Q_b\} = 0$, one has to exploit the restriction (5) and equations of motion.

The physical state of the QHS is defined as $Q_{(a)b}|phys>= 0$. This condition comes out to be consistent with the Dirac's prescription for the quantization of theories with first-class constraints [1,2]. The details of the constraints analysis has been performed in our earlier work [4] where it has been shown that the constrained field equation (5) can be incorporated in the physicality condition $Q_{(a)b}|phys>= 0$ in a subtle manner (see, e.g. [4] for details). For our present Abelian 2-form gauge theory, the BRST and anti-BRST charges play their separate and independent roles as has been established in [4] by performing a detailed constraint analyses of this theory.

5 Conclusions

In our present investigation, we have concentrated on the appropriate Lagrangian densities of the 4D free Abelain 2-form gauge theory that (i) respect the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations that were derived in our earlier works [3-5], (ii) are free of a

specific Lagrange multiplier 4D vector field which was introduced in our earlier endeavours to incorporate the analogue of the CF type restriction [3-5], (iii) are endowed with terms that carry standard meaning of the quantum field theory[6], and (iv) can be generalized so as to prove that the present 4D theory is a field theoretic model for the Hodge theory [15,16].

It is pertinent to point out that the Lagrangian densities in (6) and (7) can be recast into different simple and beautiful forms as is evident from equations (12), (13), (16) and (17). This should be contrasted, however, with the Lagrangian densities (1) and (4) which cannot be recast into the above beautiful forms because of (i) the Lagrange multiplier term (i.e. $L^\mu(B_\mu - \bar{B}_\mu - \partial_\mu \phi)$), and (ii) the kinetic term for the massless scalar field (i.e. $-(1/2)\partial_\mu \phi \partial^\mu \phi$). Thus, it is clear that the Lagrangian densities (6) and (7), that respect the same symmetry transformations as (1) and (4), are more appealing and more economical than their counterparts in (1) and (4).

The anticommutativity property of the nilpotent (anti-)BRST symmetry transformations owes its origin to the analogue of the CF condition (cf. (5), (11)) which describes a constrained surface on the 4D spacetime manifold. The key insight, for the existence of this relation, comes from the superfield approach to BRST formalism in the context of our present theory [5]. It is very interesting to note that, despite our present 4D gauge theory being an *Abelian* 2-form gauge theory, an analogue of the CF condition (which is the hallmark of a *non-Abelian* 1-form gauge theory) exists for the sanctity of the anticommutativity property of the (anti-)BRST symmetry transformations. Recently, we have been able to show the time-evolution invariance of this restriction in the Hamiltonian formalism [17].

There are a few relevant points that have to be emphasized. First, unlike non-Abelian 1-form theory [14], the above CF type restriction does not connect the auxiliary vector fields B_μ and $\bar{B}_\mu$ with any kind of (anti-)ghost fields of the theory. Rather, the above condition (5) is a relationship between the auxiliary fields and scalar field of the theory which are all bosonic in nature. Second, the analogue of the CF restriction present in our Abelian 2-form gauge theory has been shown [3] to have deep connection with the concept of gerbes. These geometrical objects, at the moment, are one of the very active areas of research in theoretical high energy physics. Finally, it would be nice to establish connection between the above fermionic (anti-)BRST charges and the twisted supercharges of the extended supersymmetry algebra. We plan to pursue the above cited issues further for our future investigations in the realm of 2-form and higher-form (non-)Abelian gauge theories [18].

[6] It will be noted that, in our previous attempts [3-5], the kinetic term for the massless scalar field turned out to possess a negative sign due to the constraint field equation.

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